TD 1

$\ln e = 1$
$\ln e^x = x$
$e^{\ln x} = x$

Exercice 1

$x(t) = e^{0.05t}$ $z(t) = e^{0.01t}$

a) $y = x$ $y(t) = x(t)$ $\frac{y(t) = e^{0.05t}}{g_y = \frac{d\ln x(t)}{dt}}$ ou $\frac{d(y(t))/dt}{y(t)}$ $(\ln u)' = \frac{u'}{u}$

$\ln y(t) = \ln e^{0.05t}$ $\ln y(t) = 0.05t$ $g_y = \frac{d\ln y(t)}{dt} = 0.05$ $g_y = 0.05$ ou 5%

b) $y(t) = z(t)$ $y(t) = e^{0.01t}$ $y' = (e^u)' = u' e^u$

$\ln(y(t)) = \ln e^{0.01t}$ $\ln y(t) = 0.01t$ $\frac{d\ln y(t)}{d(t)} = 0.01 = g_y$

c) $y = xz$ $y = e^{0.05t} e^{0.01t}$ $e^a \times e^b = e^{a+b}$ ou 6%

$y = e^{0.06t}$ $g_y = \frac{d\ln y}{dt}$ $\ln y(t) = \ln e^{0.06t}$ $\ln y(t) = 0.06t = g_y$

d) $y = x/z$ $y = \frac{e^{0.05t}}{e^{0.01t}} = e^{0.05t - 0.01t} = e^{0.04t}$ $\frac{e^a}{e^b} = e^{a-b}$

$y = e^{0.04t}$ $g_y = \frac{d\ln y(t)}{dt}$ $\ln(y(t)) = \ln e^{0.04t}$ $\ln(y(t)) = 0.04t = g_y$

e) $y(t) = x^\beta z^{1-\beta}$ avec $\beta = \frac{1}{2}$

$y(t) = x(t)^\beta \times z(t)^{1-\beta} = (e^{0.05t})^{1/2} (e^{0.01t})^{1/2} = e^{0.025t} e^{0.005t} = e^{0.025t + 0.005t}$

$\frac{d\ln y(t)}{d(t)} = g_y$ $\ln y(t) = \ln e^{0.03t}$ $\ln y(t) = 0.03t = g_y$ ou 3%

f) $y = (x/z)^\beta$ avec $\beta = 1/3$

$y = \left(\frac{x(t)}{z(t)}\right)^\beta = \left(\frac{e^{0.05t}}{e^{0.01t}}\right)^{1/3} = (e^{0.05t - 0.01t})^{1/3} = (e^{0.04t})^{1/3} = e^{0.04t/3}$

ou 1,33%

$g_y = \frac{d\ln y(t)}{dt}$ $\ln y(t) = \ln e^{0.04t/3}$ $\ln y(t) = 0.04t \times 3 = 0.013 = g_y$

Exercice 2

$\ln x^\alpha = \alpha \ln x$

a) $y = K^\beta$ $\ln y = \ln K^\beta$ $\ln y = \beta \ln K$

$\frac{d\ln y}{dt} = \beta \frac{d\ln K}{dt}$ $g_y = \beta g_K$

b) $y = K/m$ $y(t) = \frac{K(t)}{m(t)}$ $\ln y(t) = \ln\left(\frac{K(t)}{m(t)}\right)$ $\ln y(t) = \ln K(t) - \ln m(t)$

$\frac{d\ln y(t)}{dt} = \frac{d\ln K(t)}{dt} - \frac{d\ln m(t)}{dt}$ $g_y = g_K - g_m$

c) $y = \left(\frac{K}{m}\right)^\beta$ $\ln y(t) = \ln \frac{K(t)^\beta}{m(t)^\beta}$ $\ln y(t) = \beta \ln K(t) - \beta \ln m(t)$

$\frac{d\ln y(t)}{dt} = \beta \frac{d\ln K(t)}{dt} - \beta \frac{d\ln m(t)}{dt}$ $g_y = \beta g_K - \beta g_m = \beta(g_K - g_m)$

d) $y = [K\ell]^\beta \left(\frac{1}{m}\right)^{1-\beta}$ $y(t) = [K(t)\ell(t)]^\beta \left(\frac{1}{m(t)}\right)^{1-\beta}$ $\ln ab = \ln a + \ln b$

$\ln y(t) = \ln\left[(K(t)\ell(t))^\beta \left(\frac{1}{m(t)}\right)^{1-\beta}\right] = \ln(K(t)\ell(t))^\beta + \ln\left(\frac{1}{m(t)}\right)^{1-\beta}$

$\beta[\ln K(t) + \ln \ell(t)] + (1-\beta)\ln\frac{1}{m(t)} = \beta \ln k(t) + \beta \ln \ell(t) + (1-\beta)[-\ln m(t)]$

$\ln y(t) = \beta \ln k(t) + \beta \ln \ell(t) - (1-\beta)\ln m(t)$

$\frac{d\ln y(t)}{dt} = \beta \frac{d\ln k(t)}{dt} + \beta \frac{d\ln \ell(t)}{dt} - (1-\beta)\frac{d\ln m(t)}{dt}$

$g_y = \beta g_K + \beta g_\ell - (1-\beta) g_m = \beta(g_K + g_\ell) - (1-\beta) g_m$

EXERCICE 3

a) $y = xz$ on cherche la forme fonctionnelle.

$\frac{x}{x} = 0.10 \Leftrightarrow \frac{\dot{x}}{x} = g_x = \frac{d\ln x(t)}{dt} = 0.10$

$\frac{d\ln x(t)}{d(t)} = 0.10 \quad d\ln x(t) = 0.10\,dt \quad \int d\ln x(t) = \int 0.10\,dt$ $\quad \int_0^t dt = 0{,}10(t+c)$

$\ln x(t) = 0.10(t+c) \quad e^{\ln x(t)} = e^{0.10(t+c)} \quad x(t) = e^{0.10t} e^{0.10c}$

$x(0) = e^{0.10c} = 2 = e^c \quad x(t) = x(0) e^{0.10t}$ ou $x(t) = x(0) e^{gxt}$

$z(t) = z(0) e^{gzt} \quad z(t) = e^{0.20t}$

*

Calculons $y(t)$ pour $t \in \{0, 1, 2, 10\}$

$y = xz \quad y(t) = 2e^{0.10t} e^{0.20t} \quad y(t) = 2e^{0.30t} \quad y(0) = 2 \quad y(1) = 2e^{0.30}$

$y(2) = 2e^{0.30\times 2} \quad y(10) = 2e^{0.30\times 10}$

b) $y = \frac{x}{z} \quad y(t) = \frac{x(t)}{z(t)} \quad y(t) = \frac{2e^{0.10t}}{e^{0.20t}} \quad y(t) = 2e^{-0.10t} \quad t \in \{0, 1, 2, 10\}$

c) $y(t) = x(t)^\beta z(t)^{1-\beta} \quad y(t) = (2(e^{0.1t})^{1/3}(e^{0.2t})^{2/3}$

$y(t) = 2^{1/3} e^{0.1/3t} e^{0.4/3t} = 2^{1/3} e^{0.5/3t} \quad y(t) = 2^{1/3} e^{0.5/3t}$

$y(t) = 2^{1/3} e^{0.5/3t}, \quad t \in \{0, 1, 2, 10\}$

* $\frac{\dot{x}}{x} = 0.10 \quad \frac{\dot{x}}{x} = g_x = \frac{d\ln x(t)}{dt} = 0.10$

$\frac{d\ln x(t)}{dt} = 0.10 \quad d\ln x(t) = 0.10\,dt \Leftrightarrow \int \ln x(t) = \int 0.10\,dt$

$\ln x(t) = 0.10t + c \quad e^{\ln x(t)} = e^{0.10t+c} \quad x(t) = e^{0.10} e^c$

$x(0) = e^0 e^c = e^c = 2$ d'où $x(t) = 2e^{0.10t}$ ou $x(t) = x(0) e^{gxt}$

METHODE

a) $\frac{\dot{x}}{x} = \frac{d\ln x(t)}{d(t)} = 0.10 \Leftrightarrow d\ln x(t) = 0.10\,dt$

$\Leftrightarrow \int d\ln x(t) = 0.10 \int dt \Leftrightarrow \ln x(t) = 0.10(t+c)$

$\Leftrightarrow e^{\ln x(t)} = e^{0.10(t+c)} \Leftrightarrow x(t) = e^{0.10(t+c)} \Leftrightarrow x(t) = e^{0.10t} e^{0.10c}$

$\Leftrightarrow x(0) = 2 \quad x(0) = 1e^{0.10c} \quad e^{0.10c} = 2 \quad x(t) = 2e^{0.10t}$

Faire la même pour $z(t)$ $\quad z(t) = e^{0.20t} \quad y = xz \quad y = 2e^{0.10t} e^{0.20t}$

$= 2e^{0.30t}$